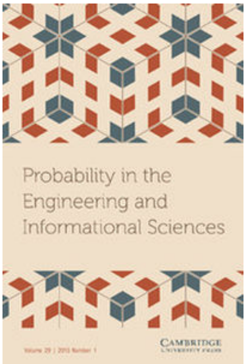

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TESTING FOR HARMONIC NEW BETTER THAN USED IN EXPECTATION

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A statistic for testing the null hypothesis that F is the exponential distribution against the alternative of harmonic new better than used in expectation (HNBUE) is proposed. The asymptotic distribution theory for this statistic is derived under the null hypothesis and asymptotic relative efficiency (ARE) with respect to other competing tests for HNBUE is evaluated. This test is applied to the leukemia data described in Bryson and Siddiqui (1969).

1. INTRODUCTION

In reliability theory various concepts of aging have been proposed to study lifetimes of systems and components. Rolski [9] proposed a new class of life distributions called the harmonic new better than used in expectation (HNBUE), with its dual, HNWUE (W for worse). Let $\bar{F} = (1 - F)$ be the survival function of a nonnegative valued random variable X. Then:

DEFINITION 1.1: *A life distribution F (i.e., a distribution with support on $[0,\infty)$ with survival function $\bar{F}$ and finite mean $\mu = \int_0^\infty \bar{F}(t)\,dt$ is said to be harmonic new better than used in expectation (HNBUE) if*

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$$\int_x^\infty \bar{F}(t)\,dt \le \mu \exp(-x/\mu), \quad \text{for every } x \ge 0$$

(and HNWUE when the above inequality is reversed for all $x \ge 0$).

Klefsjö [6] proposed some tests for HNBUE based on the following properties: If F is an HNBUE (HNWUE) life distribution with mean μ then

$$\int_0^\infty \bar{F}^\nu(x)\,dx \ge (\le)\,\frac{\mu}{\nu} \quad \text{for } \nu = 2,3,\ldots$$

and

$$\int_0^\infty \{1 - \bar{F}^\nu(x)\}\,dx \le (\ge)\,\mu \sum_{i=1}^{\nu} \frac{1}{i} \quad \text{for } \nu = 2,3,\ldots.$$

The test statistics proposed by him are

$$Q_1(\nu) = \sum_{i=1}^{n} J_1(i/n)t(i)/S_n$$

and

$$Q_2(\nu) = \sum_{i=1}^{n} J_2(i/n)t(i)/S_n$$

with $J_1(u) = -1/\nu + \nu(1-u)^{\nu-1}$ and $J_2(u) = \sum_{i=1}^{\nu}(1/i) - \nu u^{\nu-1}$.

Here $0 = t(0) \le t(1) \le t(2) \le \cdots \le t(n)$ is an ordered sample from the continuous distribution F and $S_i = \sum_1^i D_j$ for $i = 1,2,\ldots,n$, where

$$D_f = (n-j+1)(t(j) - t(j-1)) \quad \text{for } j = 1,2,\ldots,n$$

denote the normalized spacings. Recently Aly [1] derived a similar test statistic, A_n, which is based on the measure of departure from H_0 given by

$$\Delta(t) = \int_0^{F^{-1}(t)} (\bar{F}(x) - e^{-x/\mu})\,dx = H(t)/H(1) + \exp(-F^{-1}(t)/\mu) - 1,$$

where $H(t) = \int_0^{F^{-1}(t)} \bar{F}(x)\,dx$ is the total time on test (TTT) transform corresponding to F.

Section 2 investigates a test statistic for testing exponentiality versus HNBUE (HNWUE). In Section 3, the asymptotic relative efficiency (ARE) of this test is compared with respect to other tests for HNBUE. In Section 4 we apply these results to the leukemia data reported in Bryson and Siddiqui [3].

2. A TEST FOR HNBUE

In this section we study the problem of testing

$$\mathrm{H}_0\colon F(x) = F_0(x) = 1 - \exp(-x/\mu) \quad \text{for every } x \geq 0, \mu > 0 \ (\mu \text{ unspecified})$$

against the alternative:

$$\mathrm{H}_1\colon F \text{ has HNBUE property and is not exponential,}$$

based on a random sample from the life distribution F. The test is motivated by considering the following integral as a measure of deviation from H_0 for a given F. Let

$$\nabla_F = \int_0^\infty \int_x^\infty [\bar{F}(t) - \bar{F}_0(t)]\, dt\, dF_0(x) = \int_0^\infty D_F(x)\, dF_0(x), \tag{2.1}$$

where

$$D_F(x) = \int_x^\infty [\bar{F}(t) - \bar{F}_0(t)]\, dt, \quad \text{for } x \geq 0. \tag{2.2}$$

Note that $D_F(x) = 0$ for every $x \geq 0$ if and only if H_0 is true, while it is negative for HNBUE distributions that are different from the exponential. $D_F(x)$ is a measure of the deviation from H_0 toward H_1, and ∇_F is an average value of this deviation. Large negative values of ∇_F indicate HNBUE property.

The sample analog of the parameter ∇_F forms the basis for our proposed test. Substituting the empirical survival function $\bar{F}_n$ for $\bar{F}$ in (2.1), we obtain the statistic

$$T_n = \int_0^\infty \int_x^\infty [\bar{F}_n(t) - \bar{F}_0(t)]\, dt\, dF_0(x). \tag{2.3}$$

The asymptotic normality of T_n is established in Theorem 2.1, whose proof depends on Lemmas 2.2 and 2.3. Throughout, we make the assumption that F is a continuous map from positive reals to the unit interval, with a continuous inverse.

THEOREM 2.1: *Under H_0, the limiting distribution of $n^{1/2}(T_n - \nabla_F)$ is an $N(0,\sigma^2)$ where*

$$\sigma^2 = \int_0^1 \int_0^1 \int_u^1 \int_v^1 [\min(s,t) - st]\, dF^{-1}(s)\, dF^{-1}(t)\, du\, dv. \tag{2.4}$$

Let $u = F(x)$ and $B_n(u) = n^{1/2}(\bar{F}_n(F^{-1}(u)) - (1 - u))$. It is well known that $B_n(u)$ converges in distribution to the standard Brownian bridge $B^0(u)$ with mean zero and covariance function $k(u,v) = \min(u,v) - uv$, $0 \leq u,v \leq 1$. Then, under H_0, the sample version of $D_F(F^{-1}(u))$ can be written as follows:

$$D_n(u) = \int_{F^{-1}(u)}^{\infty} [\bar{F}_n(t) - \bar{F}(t)]\, dt$$
$$= \int_u^1 [\bar{F}_n(F^{-1}(s)) - (1 - s)]\, dF^{-1}(s).$$

For the proof of Theorem 2.1 we require the following lemmas:

LEMMA 2.2: *If $D_n = \{D_n(u): 0 \le u \le 1\}$, then*

$$n^{1/2} D_n \to D^0 \quad \text{in distribution as } n \to \infty,$$

where

$$D^0(u) = \int_u^1 B^0(s)\, dF^{-1}(s), \qquad 0 \le u \le 1.$$

PROOF: To prove that

$$n^{1/2} D_n(u) \to \int_u^1 B^0(s)\, dF^{-1}(s)$$

in distribution, for every fixed u, $0 \le u \le 1$, it is sufficient to show that the function

$$h(B_n(.)) = \int_u^1 B_n(s)\, dF^{-1}(s)$$

is continuous (or a.e. continuous) in the Skorohod metric d_s for all $B^0(s) \in D$, so that the result follows by the continuous mapping theorem (see Billingsley [2, p. 30]). Since $F^{-1}(s)$ is continuous for $0 \le s \le 1$, the result $n^{1/2} D_n \to D^0$ will then follow if we can show that $\int_u^1 B^0(s)\, dF^{-1}(s)$ as a functional of B^0 is continuous in d_s.

Now for any sequence of functions B_k^0 converging to B^0 in d_s, there exist functions λ_k such that $\lim_{k\to\infty} B_k^0(\lambda_k(u)) = B^0(u)$ uniformly in u and $\lambda_k(u) = u$ uniformly in u (Skorohod topology: see [2] for details). As every element of D is bounded and has at most a countable number of discontinuities, it is Riemann integrable.

Let $P_h: 0 < u_1 < \cdots < u_h < s$ and $P_h': 0 < \lambda_k(u_1) < \cdots < \lambda_k(u_h) < s$ be two sequences of partitions such that P_h is for $\int_u^1 B^0(s)\, dF^{-1}(s)$ and P_h' is for $\int_u^1 B_k^0(s)\, dF^{-1}(s)$. As $k \to \infty$, the upper and lower sums for the latter integral converge to those for the former. Therefore, it follows that

$$\int_u^1 B_k^0(s)\, dF^{-1}(s) \to \int_u^1 B^0(s)\, dF^{-1}(s), \qquad 0 \le u \le 1$$

and that $\int_u^1 B^0(s)\, dF^{-1}(s)$ is continuous in d_s. ■

LEMMA 2.3: *The stochastic integral $\int_0^1 D^0(u)\, du$ exists in the sense of convergence in mean square and has an $N(0, \sigma^2)$ distribution, where σ^2 is given by* (2.4).

PROOF: As a first step, define the stochastic integral

$$D^0 = \int_u^1 B^0(s)\, dF^{-1}(s),$$

where $\{B^0(s): 0 \le s \le 1\}$ is a Brownian bridge. Let

$$0 = u_{p,0} < u_{p,1} < \cdots < u_{p,p} = u$$

such that

$$\lim_{k\to\infty} \max_{1\le k\le p} (u_{p,k} - u_{p,k-1}) = 0.$$

We form the "Riemann sums"

$$S_p = \sum_{k=1}^{p} B^0(s^*_{p,k})(s_{p,k} - s_{p,k-1}),$$

while

$$S_p S_q = \sum_{k=1}^{p}\sum_{j=1}^{q} B^0(s^*_{p,k})B^0(t^*_{q,j})(s_{p,k} - s_{p,k-1})(t_{q,j} - t_{q,j-1}),$$

where $s_{p,k-1} \le s^*_{p,k} \le s^k_p$ and $t_{q,j-1} \le t^*_{q,j} \le t_{q,j}$. Then

$$\mathrm{E}(S_p S_q) = \sum\sum [\min(s^*_{p,k}, t^*_{q,j}) - s^*_{p,k}t^*_{q,j}](s_{p,k} - s_{p,k-1})(t_{q,j} - t_{q,j-1})$$

and we obtain

$$\lim_{p,q\to\infty} \mathrm{E}(S_p S_q) = \int_u^1 \int_v^1 [\min(s,t)\text{-}st]\, ds\, dt.$$

This limit is the same for all sequence of subdivisions and all choices of the intermediate points. Now appealing to Loeve's Theorem 6.1.5 [7, p. 469], we see that the "Riemann sum" S_p converges in mean square. Thus, the stochastic integral $D^0(u)$ exists in the sense of convergence in mean square. It follows from the fact that the Brownian bridge is a Gaussian process that $\{D^0(u): 0 \le u \le 1\}$ is also Gaussian. Its distribution is characterized by its mean value and covariance function:

$$\mathrm{E}(D^0(u)) = \int_u^1 \mathrm{E}B^0(s)\, dF^{-1}(s) = 0 \quad \text{for all } u$$

and

$$k'(u,v) = \mathrm{Cov}(D^0(u), D^0(v)) = \int_u^1 \int_v^1 [\min(s,t)\text{-}st]\, dF^{-1}(s)\, dF^{-1}(t). \tag{2.5}$$

In a similar way, the integral $\int_0^1 \int_0^1 k'(u,v)\, du\, dv$ exists because $k'(u,v)$ is continuous on I^2. Thus, the integral $\int_0^1 D^0(u)\, du$ is well defined as a limit in mean square of the usual approximating sums. It can be shown that this integral defines a random variable. As the approximating sum of this integral is the sum of independently and normally distributed random variables we see that $\int_0^1 D^0(u)\,du$ is also normally distributed. Moreover, it is easily seen that

$$\mathrm{E}\left[\int_0^1 D^0(u)\,du\right] = 0$$

while

$$\mathrm{Var}\left[\int_0^1 D^0(u)\,du\right] = \int_0^1\int_0^1 k'(u,v)\,du\,dv,$$

where $k'(u,v)$ is given by (2.5). ■

Now, by Lemmas 2.2 and 2.3, $n^{1/2}T_n = \int_0^1\int_u^1 B_n(s)\,dF^{-1}(s)\,du$ has a limiting $N(0,\sigma^2)$, with σ^2 given by (2.4) which proves Theorem 2.1.

However, the distribution of $n^{1/2}T_n$ is not scale invariant; that is, it depends on the scale parameter μ. In order to make our test scale invariant, we will use the test statistic, $n^{1/2}(T_n/\bar{X}_n)$, where $\bar{X}_n$ is the sample mean. Then, under H_0, the limiting distribution of $n^{1/2}(T_n/\bar{X}_n)$ is $N(0,\sigma^2/\mu^2)$ using the result of Theorem 2.1 and Slutsky's theorem. The form (2.3) of the statistic is not very convenient for calculation. For computational purposes, $n^{1/2}(T_n/\bar{X}_n)$ can be rewritten as

$$n^{1/2}(T_n/\bar{X}_n) = n^{-1/2}\sum_{i=1}^{n}\exp(-X_i/\bar{X}_n) - \frac{1}{2} = n^{-1/2}\sum_{i=1}^{n}\exp(-nD_i) - \frac{1}{2},$$

where D_i's are the uniform spacings. Thus, our test statistic is of the form

$$n^{1/2}(T_n/\bar{X}_n) = n^{-1/2}\sum_{i=1}^{n} h(nD_i),$$

for some function $h(.)$. It is interesting to note that spacings statistics of this form, when $h(.)$ satisfies some mild regularity conditions, have been shown to be asymptotically normal (see Rao and Sethuraman [8]) by a completely different approach.

When doing calculations under H_0, we have $\sigma^2/\mu^2 = 1/48$. Therefore, for large n, the null distribution of $(48n)^{1/2}T_n/\bar{X}_n$ is approximately standard normal. Although the rate of convergence to normality is an important issue, we do not investigate that question here. To test H_0 against H_1 (HNBUE) at approximate α level, reject H_0 in favor of H_1 if $(48n)^{1/2}T_n/\bar{X}_n \le -z_\alpha$, where z_α is the upper percentile point of a standard normal distribution. Similarly, to test H_0 against H_1 (HNWUE) at the approximate α level, in large samples, reject H_0 in favor of H_1 if $(48n)^{1/2}T_n/\bar{X}_n > z_\alpha$.

3. PITMAN ASYMPTOTIC RELATIVE EFFICIENCY

We next compare the T_n test with respect to Klefsjö's [6] tests Q_1 and Q_2 on the basis of Pitman ARE. Consider a sequence of alternative distributions indexed by the parameters θ_n, where $\theta_n = \theta_0 + (c/\sqrt{n})$, c is an arbitrary positive constant, and θ_0

corresponds to the exponential distribution. Then the ARE of T_1 with respect to T_2 can be shown for such smooth alternatives to be

$$e_F(T_1,T_2) = \lim_{n\to\infty} \left[\frac{\left.\dfrac{d}{d\theta} E_\theta(T_1)\right|_{\theta=\theta_0}}{\left.\dfrac{d}{d\theta} E_\theta(T_2)\right|_{\theta=\theta_0}} \right]^2 \frac{\sigma_2^2(\theta_0)}{\sigma_1^2(\theta_0)}, \tag{3.1}$$

where $\sigma_1^2(\theta_0)$ and $\sigma_2^2(\theta_0)$ are null asymptotic variances of T_1 and T_2, respectively (see, e.g., Fraser [4, Ch. 7.3]. For the computations of ARE we consider the Weibull family of alternatives given by

$$F_\theta(x) = 1 - \exp(-x^\theta) \quad \text{for } x \geq 0, \theta \geq 1.$$

Direct calculations give the following AREs:

$$\begin{aligned} e_F(T_n,Q_1(2)) &= 1.04 & e_F(T_n,Q_2(2)) &= 1.12 \\ e_F(T_n,Q_1(3)) &= 0.99 & e_F(T_n,Q_2(3)) &= 1.20 \\ e_F(T_n,Q_1(4)) &= 1.00 & e_F(T_n,Q_2(4)) &= 1.28 \\ e_F(T_n,Q_1(5)) &= 1.03 & e_F(T_n,Q_2(5)) &= 1.36. \end{aligned}$$

Here the efficacies of the tests $Q_1(3)$ and $Q_2(2)$ are maximums for Weibull alternatives from the class of tests $Q_1(\nu)$; $\nu = 2,3,\ldots$ and $Q_2(\nu)$; $\nu = 2,3,\ldots$ of Klefsjö, respectively. The T_n test also compares favorably with Aly's test, A_n, since Q and A_n tests are known to be equally efficient [1]. Similar ARE computations are possible for several other alternative models to exponentiality.

4. A PRACTICAL EXAMPLE

Bryson and Siddiqui [3] have analyzed data that are survival times, in days from diagnosis, of 43 patients suffering from chronic granulocytic leukemia (Table 1).

Table 1. The Ordered Survival Times (in days from diagnosis)

7	429	579	968	1877
47	440	581	1077	1886
58	445	650	1109	2045
74	455	702	1314	2056
177	468	715	1334	2260
232	495	779	1367	2429
273	497	881	1534	2509
285	532	900	1712	
317	571	930	1784	

Based on this data set, the problems of testing exponentiality against the property of decreasing mean residual life (DMRL) and the property of new better than used in expectation (NBUE) were considered by Hollander and Proschan [5]. The resulting p values of 0.08 and 0.05, respectively, based on their tests, fail to reject exponentiality. Aly [1] continued the analysis of this leukemia data by considering a nonparametric graphical technique (HNBUE-plot). Figure 2 of Aly [1] of the HNBUE plot suggests that the data are coming from an HNBUE distribution.

Next, we use the statistic T_n of (2.3) to test exponentiality against HNBUE property. We find $(48n)^{1/2}T_n/\overline{X}_n = -1.739$ with a corresponding p value of 0.041. Thus the test supports an HNBUE property which conforms to the graphical conclusion of Aly [1].

References

1. Aly, E.-E.A.A. (1992). On testing exponentiality against HNBUE alternatives. *Statistics and Decisions* 10: 239–250.
2. Billingsley, P. (1968). *Convergence of probability measures.* New York: Wiley.
3. Bryson, M.C. & Siddiqui, M.M. (1969). Some criteria for aging. *Journal of the American Statistical Association* 64: 1472–1483.
4. Fraser, D.A.S. (1975). *Nonparametric methods in statistics.* New York: Wiley.
5. Hollander, M. & Proschan, F. (1975). Tests for the mean residual life. *Biometrika* 62: 585–593.
6. Klefsjö, B. (1983). Testing exponentiality against HNBUE. *Scandinavian Journal of Statistics* 10: 65–75.
7. Loeve, M. (1963). *Probability theory.* Princeton: D. Van Nostrand Co.
8. Rao, J.S. & Sethuraman, J. (1975). Weak convergence of empirical distribution functions of random variables subject to perturbations and scale factors. *Annals of Statistics* 3: 299–313.
9. Rolski, T. (1975). Mean residual life. *Bulletin of the International Statistical Institute* 46: 266–270.